NOTATION

T, stress force tensor; **M**, moment stress tensor; **u**, displacement vector; **ω**, rotation vector; **X**, external mass force vector; Y, external mass moment vector; ρ, density; I, tensor characterizing the inertial properties of the medium during rotation; H, internal energy density; F, free energy density; S, internal entropy density; K, kinetic energy; A, power of the external mechanical forces; Q, power of the external thermal sources; q, thermal flux vector; w, heat liberation density; Θ, absolute temperature; Θ_0, initial temperature; E, unit tensor; a^B, vector accompanying the tensor **B**; $\mathbf{B}^+$, symmetric part of the tensor **B**; $\mathbf{B}^-$, antisymmetric part of the tensor B; $\bar{f}$, Laplace transform in f.

LITERATURE CITED

1. V. A. Pal'mov, "Fundamental equations of nonsymmetric elasticity theory," Prikl. Mat. Mekh., 28, No. 3, 401-408 (1964).
2. E. V. Kuvshinskii and É. P. Aero, "Continual theory of asymmetric elasticity. Taking account of internal rotation," Fiz. Tverd. Tela, 5, No. 9, 2591-2598 (1963).
3. W. Nowacki, "Couple-stresses in the theory of thermoelasticity," Bull. Acad. Polon. Sci., Ser. Sci. Technol., 14, No. 8, 505-513 (1966).
4. Vl. N. Smirnov, "Equations of generalized thermoelasticity of a Cosserat medium," Inzh.-Fiz. Zh., 39, No. 4, 716-723 (1980).
5. V. A. Pal'mov, Vibrations of Elastic-Plastic Solids [in Russian], Nauka, Moscow (1976).
6. A. I. Lur'e, Nonlinear Elasticity Theory [in Russian], Nauka, Moscow (1980).
7. W. Nowacki, Elasticity Theory [Russian translation], Mir, Moscow (1975).
8. H. Parkus, Unsteady Temperature Stresses [in Russian], Gos. Izd-vo Fiz.-Mat. Lit., Moscow (1963).

EQUIVALENCE OF CERTAIN TYPES OF RHEOLOGICAL EQUATIONS OF STATE FOR POLYMER MEDIA

PART 1. GENERAL ANALYSIS

B. M. Khusid

UDC 532.135

The conditions are established under which rheological relaxation equations and rheological integral equations will be equivalent.

According to the classification proposed by C. Truesdell and W. Noll [1, 2], the rheological equations of state for aftereffect media fall into three groups: differential equations, relaxation (or strain rate) equations, and integral equations. Equations of the differential type are applicable only to flow with a small Deborah number, i.e., to fluids with a relaxation time much shorter than the time scale of flow. In other cases one must use either relaxation equations or integral equations of state. Many relaxation equations and integral equations of state have already been proposed [4-7]. As a rule, they are partly based on microscopic models of polymer fluids and on certain assumptions regarding the motion of the medium. They also include several parameters which must be evaluated empirically for any specific material. The rheological equations more or less agree with experiments. According to the bibliography on this subject, however, none of them adequately describes the rheological characteristics of various fluids in complex transient strain states. This makes it necessary to try various models for a given material and then, after comparison with the experiment, select the most applicable ones. Such a diversity of rheological equations for fluids with memory impedes the programming of numerical solution of hydrodynamic and thermal problems for rheologically complex fluids. In the case of relaxation equations of state one formulates the problems of hydrodynamics and heat transfer in the form of partial differential equations, which can be solved by conventional methods of finite differences. For integral

Belorussian Polytechnic Institute, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 42, No. 4, pp. 670-677, April, 1982. Original article submitted October 10, 1981.

0022-0841/82/4204-0470$07.50

equations of state these problems are formulated in terms of systems of integrodifferential equations. Here it becomes necessary to integrate over trajectories of moving fluid particles, which makes it difficult to apply conventional methods of finite differences. In view of this, there arises the question as to whether integral equations of state can be reduced to equivalent systems of differential equations. A reduction of various rheological equations to a single type would also reveal the more distinct differences between the physical hypotheses on the basis of which various rheological equations of state have been constructed for aftereffect media. In this study will be considered this problem of reducing the integral rheological equations of state of the (1) kind for an incompressible fluid to an equivalent system of differential equations not containing higher than first derivatives of the stress tensor with respect to time

$$\mathbf{T}=\sum_{\alpha,\beta}\mathbf{T}_{\alpha\beta},\quad \mathbf{T}_{\alpha\beta}=\int_{-\infty}^{t}\mu_\alpha(t,\ \tau)\,[\mathbf{\Omega}_\beta(t,\ \tau)-\mathbf{E}]\,d\tau. \tag{1}$$

Here summation is performed over different pairs (α, β) and for different β's, moreover, the sets of corresponding functions μ_α can also differ. In this equation the isotropic additive tensor has been omitted, μ_α is the relaxation function, and $\mathbf{E}$ is the unit tensor. Integration is performed over the trajectory of a moving fluid particle. Most of the now available semiempirical integral equations of state for polymer melts and concentrates are put in the (1) form [4-7]. In analogy with the theory of relaxing lattice according to Lodge, Yamamoto, et al. [8], one can interpret expression (1) as one describing the "resultant" tension $T_{\alpha\beta}$ on lattices of kind β with nodes of group α. Accordingly, the quantities $\mu_\alpha(t, \tau)$ are proportional to the number of α nodes formed at instant of time τ and still existing at instant of time t, while the symmetric tensors $\Omega_\beta(t, \tau)$ − $\mathbf{E}$ represent strain produced by flow of the fluid in a β-lattice at instant of time τ relative to its state at instant of time t, with $\Omega_\beta(t, t) = \mathrm{E}$. We will now transform each term in sum (1) into a system of differential equations. An evaluation of the substantive derivative with respect to time yields

$$\frac{D\mathbf{T}_{\alpha\beta}}{Dt}=\int_{-\infty}^{t}\frac{D\mu_\alpha(t,\ \tau)}{Dt}\,[\mathbf{\Omega}_\beta(t,\ \tau)-\mathbf{E}]\,d\tau+\int_{-\infty}^{t}\mu_\alpha(t,\ \tau)\,\frac{D\mathbf{\Omega}_\beta(t,\ \tau)}{Dt}\,d\tau. \tag{2}$$

Equations (1) and (2) reduce to a system of first-order differential equations in time, if

$$\frac{D\mathbf{\Omega}_\beta(t,\ \tau)}{Dt}=-\sum_{\gamma}[\mathbf{A}^{\mathrm{T}}_{\gamma\beta}(t)\,\mathbf{\Omega}_\gamma(t,\ \tau)+\mathbf{\Omega}_\gamma(t,\ \tau)\,\mathbf{A}_{\gamma\beta}(t)], \tag{3}$$

$$\frac{D\mu_\alpha(t,\ \tau)}{Dt}=-\sum_{\gamma}\varkappa_{\alpha\gamma}(t)\,\mu_\gamma(t,\ \tau). \tag{4}$$

Expression (3) takes into account the symmetry of the Ω_β-tensor, superscript T denoting a transposition. In accordance with the invariance of the rheological Eq. (1) (principle of objectivity) [1, 2], a change of the reference system ($\mathbf{Q}(t)$ representing an arbitrary orthogonal tensor) $\vec{X}\to\vec{X}^*=\mathbf{Y}(t)+\mathbf{Q}(t)(\vec{X}-\vec{Z})$ will transform the tensor $\mathbf{\Omega}_\beta(t, \tau)$ into

$$\mathbf{\Omega}_\beta(t,\ \tau)\to\mathbf{\Omega}^*_\beta(t,\ \tau)=\mathbf{Q}(t)\,\mathbf{\Omega}_\beta(t,\ \tau)\,\mathbf{Q}^{\mathrm{T}}(t). \tag{5}$$

Rewriting Eq. (3) in the new reference system, with the aid of relation (5), we obtain the transformation law for the tensor $\mathbf{A}_{\beta\gamma}$ ($\delta_{\beta\gamma}$ is the Kronecker delta, $\delta_{\beta\gamma}=0$ for $\beta\neq\gamma$ and $\delta_{\beta\gamma}=1$ for $\beta=\gamma$)

$$\mathbf{A}_{\beta\gamma}\to\mathbf{A}^*_{\beta\gamma}=\mathbf{Q}\mathbf{A}_{\beta\gamma}\mathbf{Q}^{\mathrm{T}}+\frac{D\mathbf{Q}}{Dt}\mathbf{Q}^{\mathrm{T}}. \tag{6}$$

With the aid of the transformation law for the vorticity tensor $\mathrm{W}=\frac{1}{2}(\nabla\vec{v}-\nabla\vec{v}^{\mathrm{T}})$ [1, 2], namely $\mathrm{W}\to\mathrm{W}^*=\mathbf{QWQ}^{\mathrm{T}}+\mathrm{D}\mathbf{Q}/\mathrm{Dt}\,\mathbf{Q}^{\mathrm{T}}$, it is possible to represent the tensor $\mathrm{A}_{\beta\gamma}$ in the form

$$\mathbf{A}_{\beta\gamma}=\delta_{\beta\gamma}\mathbf{W}+\mathbf{B}_{\beta\gamma},\quad \mathbf{B}_{\beta\gamma}\to\mathbf{B}^*_{\beta\gamma}=\mathbf{Q}\mathbf{B}_{\beta\gamma}\mathbf{Q}^{\mathrm{T}}, \tag{7}$$

where the tensor $\mathbf{B}_{\beta\gamma}$ satisfies the principle of objectivity (6). Since the coefficients in system (4) are not functions of τ, one can express the function $\mu_\alpha(t, \tau)$ as $\mu_\alpha(t, \tau)=\sum_\gamma \mathrm{F}_{\alpha\gamma}(t, \tau)\,\varphi_\gamma(\tau)$ with $\mathrm{F}_{\alpha\gamma}$ denoting the fundamental solution to system (4) which satisfies the condition $\mathrm{F}_{\alpha\gamma}(\tau, \tau)=\delta_{\alpha\gamma}$. Inserting expressions (3), (4), and (7) into Eq. (2), and introducing the function $p_\alpha=\int_{-\infty}^{t}\mu_\alpha(t,\ \tau)\,d\tau$, we arrive at the system of first-order differential equations in time which is equivalent to Eq. (1)

$$\frac{Dp_\alpha}{Dt}+\sum_{\gamma}\varkappa_{\alpha\gamma}p_\gamma=-\varphi_\alpha, \tag{8}$$

$$\overset{0}{\mathbf{T}}_{\alpha\beta} + \sum_{\gamma} \varkappa_{\alpha\gamma}\mathbf{T}_{\gamma\beta} + \sum_{\gamma}(\mathbf{B}^{\mathrm{T}}_{\gamma\beta}\mathbf{T}_{\alpha\gamma} + \mathbf{T}_{\alpha\gamma}\mathbf{B}_{\gamma\beta}) = p_{\alpha}\sum_{\gamma}(\mathbf{B}^{\mathrm{T}}_{\beta\gamma} + \mathbf{B}_{\beta\gamma}).$$

Here the notation of the Jaumann derivatice has been used

$$\overset{0}{\mathbf{T}}_{\alpha\beta} = \frac{D\mathbf{T}_{\alpha\beta}}{DT} - \mathbf{W}\mathbf{T}_{\alpha\beta} + \mathbf{T}_{\alpha\beta}\mathbf{W}.$$

All quantities in Eqs. (8) are to be evaluated at a given point at instant of time t. These equations constitute, of course, a special case of the general form of relaxation equations for fluids which satisfy the principle of objectivity [1]. For most rheological models found in the bibliography on this subject [4-7],

$$\varkappa_{\alpha\gamma} = \varkappa_{\alpha}\delta_{\alpha\gamma}, \quad \mathbf{A}_{\gamma\beta} = \mathbf{A}_{\beta}\delta_{\gamma\beta} \tag{9}$$

in Eqs. (3) and (4). According to the interpretation based on the theory of lattice relaxation, the first of relations (9) means that the probability of breakup of group α nodes is proportional to the number of such nodes. In this case $\mu_{\alpha}(t, \tau) = \varphi_{\alpha}(\tau)\exp[-\int_{\tau}^{t}\varkappa_{\alpha}(\xi)d\xi]$, where $\varkappa_{\alpha}$ defines the probability of breakup of one node. With the second of relations (9) taken into account, Eq. (3) becomes

$$\frac{D\Omega_{\beta}(t, \tau)}{Dt} = -\mathbf{A}^{\mathrm{T}}_{\beta}(t)\,\Omega_{\beta}(t, \tau) - \Omega_{\beta}(t, \tau)\,\mathbf{A}_{\beta}(t), \quad \Omega_{\beta}|_{t=\tau} = E. \tag{10}$$

It follows from Eq. (10) (tr denotes the trace of a tensor) that $\det \Omega_{\beta}(t, \tau) = \exp\left[-\int_{\tau}^{t} \mathrm{tr}\,(\mathbf{A}^{\mathrm{T}}_{\beta} + \mathbf{A}_{\beta})\,d\xi\right]$. Since the eigenvalues of the symmetric tensor $\Omega_{\beta}(t, \tau)$ at $t = \tau$ are equal to unity and since, $\det \Omega_{\beta}(t, \tau) > 0$, these eigenvalues are always positive and there exists a positive-definite tensor $\sqrt{\Omega_{\beta}(t, \tau)}$ [9]. This permits us to use the representation

$$\Omega_{\beta} = \omega^{\mathrm{T}}_{\beta}\omega_{\beta}, \tag{11}$$

where $\omega_{\beta} = \mathbf{R}_{\beta}\sqrt{\Omega_{\beta}}$, and $\mathbf{R}_{\beta}$ is some orthogonal tensor. Inserting the representation (11) into Eq. (10) yields

$$\frac{D\omega_{\beta}(t, \tau)}{Dt} = -\omega_{\beta}(t, \tau)\,\mathbf{A}_{\beta}(t), \quad \omega_{\beta}|_{t=\tau} = \mathbf{E}. \tag{12}$$

Equations (10) and (12), which have been derived from expression (1), determine the tensors ω_{β} and Ω_{β} only at $t \geq \tau$. We will now establish several properties of the tensor $\omega_{\beta}(t, \tau)$. We assume a given mode of deformation of an $\mathbf{A}_{\beta}(t)$ lattice. Let us consider an instant of time $\tau \geq \xi$. Since $\omega_{\beta}^{-1}(\tau, \xi)\,\omega_{\beta}(t, \xi)|_{t=\tau} = \mathbf{E}$, hence the linearity of relation (12) makes

$$\omega_{\beta}^{-1}(\tau, \xi)\,\omega_{\beta}(t, \xi) = \omega_{\beta}(t, \tau), \quad t \geqslant \tau \geqslant \xi. \tag{13}$$

Differentiating this equations with respect to τ and using relation (12), we obtain

$$\frac{D\omega_{\beta}(t, \tau)}{D\tau} = \mathbf{A}_{\beta}(\tau)\,\omega_{\beta}(t, \tau), \quad \omega_{\beta}|_{\tau=t} = E. \tag{14}$$

We next define the tensor $\omega_{\beta}(t, \tau)$ at $t \leq \tau$ by the equality

$$\omega_{\beta}(t, \tau) = \omega_{\beta}^{-1}(\tau, t), \quad \tau \geqslant t. \tag{15}$$

Differentiating expression (15) with respect to τ, with relation (12) taken into account, we again arrive at Eq. (14). Therefore, Eq. (14) defines the tensor $\omega_{\beta}(t, \tau)$ at all instants t and τ. Since $\omega_{\beta}(t, \tau)\,\omega_{\beta}^{-1}(t, t_0)|_{\tau=t_0} = \mathbf{E}$ at any t_0, hence the linearity of expression (14) makes

$$\omega_{\beta}(t, \tau) = \omega_{\beta}(t_0, \tau)\,\omega_{\beta}(t, t_0). \tag{16}$$

This expression extends relation (13) to three arbitrary instants of time. Tensor $\omega_{\beta}(t, \tau)$ is the matricant [9] of Eq. (14) and relation (16) describes its basic properties. According to relation (16), the distortions of a lattice at instant of time τ relative to its state at instant of time t do not depend on the intermediate state at instant of time t_0. Property (16) fully defines the tensor $\omega_{\beta}(t, \tau)$. Starting with it, one can easily derive relation (15) and then Eqs. (12), (14), (10). Thus the second of relations (9) is equivalent to an assumption that the relative deformation of a lattice has the multiplicative property (16). On the basis of Eq. (14), it is easy to demonstrate that changing the reference system will transform the tensor $\omega_{\beta}(t, \tau)$ as follows: $\omega_{\beta}(t, \tau) \to \omega_{\beta}^{*}(t, \tau) = \mathbf{Q}(\tau)\omega_{\beta}(t, \tau)\mathbf{Q}^{\mathrm{T}}(t)$. In another study (10) the tensor $\omega_{\beta}(t, \tau)$ satisfying Eq. (14) and the tensor $\Omega_{\beta}(t, \tau)$ defined by relation (11) are called nonholonomic or generalized deformations. Relations (9) greatly simplify the rheological equations (8) and reduce them to the form of a Maxwell fluid

$$\overset{0}{\mathbf{T}}_{\alpha\beta}+\varkappa_\alpha\mathbf{T}_{\alpha\beta}+\mathbf{B}_\beta^{\mathrm{T}}\mathbf{T}_{\alpha\beta}+\mathbf{T}_{\alpha\beta}\mathbf{B}_\beta=p_\alpha(\mathbf{B}_\beta^{\mathrm{T}}+\mathbf{B}_\beta), \tag{17'}$$

$$\frac{Dp_\alpha}{Dt}+\varkappa_\alpha p_\alpha=-\varphi_\alpha. \tag{17''}$$

Such are the differential equations to which all integral rheological equations of the form

$$\mathbf{T}=\sum_{\alpha,\beta}\int_{-\infty}^{t}\varphi_\alpha(\tau)\exp\left[-\int_{\tau}^{t}\varkappa_\alpha(\xi)\,d\xi\right][\boldsymbol{\omega}_\beta^{\mathrm{T}}(t,\ \tau)\,\boldsymbol{\omega}_\beta(t,\ \tau)-\mathbf{E}] \tag{18}$$

with the tensor $\omega_\beta(t,\tau)$ satisfying the property (16) can be reduced. The quantities φ_α, $\varkappa_\alpha$ depend on the deformation mode at a given point at instant of time t. Some rheological models found in the bibliography on this subject correspond to the integral equation

$$\mathbf{T}=-\sum_{\alpha,\beta}\int_{-\infty}^{t}\tilde{\varphi}_\alpha(\tau)\exp\left[-\int_{\tau}^{t}\varkappa_\alpha(\xi)\,d\xi\right]\frac{D\Omega_\beta(t,\ \tau)}{D\tau}\,d\tau, \tag{19}$$

which is obtained through integration of Eq. (18) by parts. Functions $\tilde{\varphi}_\alpha$ and φ_α are related through the equation $\frac{D\tilde{\varphi}_\alpha}{Dt}+\varkappa_\alpha\tilde{\varphi}_\alpha=\varphi_\alpha$. Inserting this relation into Eq. (17'') yields $p_\alpha=-\tilde{\varphi}_\alpha$. A feature distinguishing model (18) from model (19) is that the quantity p_α appearing as the shear modulus in Eq. (17') can be determined from Eq. (17'') and depends on deformation history. In model (19), on the other hand, the quantity p_α depends only on the flow characteristics at instant of time t. When $\varkappa_\alpha$, φ_α, $\tilde{\varphi}_\alpha$ are constants, then this difference between the two models (18), (19) vanishes and they become fully equivalent. Rheological equations of the one (18) kind are generalizations of earlier Lodge, Ward, Jenkins models and rheological equations of the (19) kind are generalizations of earlier Oldroyd, Friedrichson, Walters models [4-7]. In order for Eqs. (17') and (17'') to merge into the single relation (18), it is necessary that the initial conditions for them be stipulated in the quiescent state, i.e., at $t\to\infty$ (the problem of initial conditions for relaxation equations has been treated more thoroughly in [3]). The tensor $\mathbf{B}_\beta$ in Eqs. (17') and (17'') has been defined ambiguously. A transformation $\mathbf{B}_\beta\to\tilde{\mathbf{B}}_\beta=\mathbf{B}_\beta-b_\beta\mathbf{E}-\mathbf{R}_\beta(\mathbf{T}_{\alpha\beta}-p_\alpha\mathbf{E})$, $p_\alpha\to\tilde{p}_\alpha=p_\alpha-\xi_{\alpha\beta}$, $\varkappa_\alpha\to\tilde{\varkappa}_\alpha=\varkappa_\alpha+2b_\beta$, $\mathbf{T}_{\alpha\beta}\to\tilde{\mathbf{T}}_{\alpha\beta}=\mathbf{T}_{\alpha\beta}-\xi_{\alpha\beta}\mathbf{E}$ where $\frac{D\xi_{\alpha\beta}}{Dt}+\tilde{\varkappa}_\alpha\xi_{\alpha\beta}=2b_\beta p_\alpha$ and $\mathbf{R}_\beta$ is an arbitrary antisymmetric tensor, does not have the form of Eqs. (17') and (17''). Since the stress tensor has been defined precisely, except for the isotropic tensor, this transformation will not alter the rheological equation. In order for the integral equation (1) to be reducible to the system (17'), (17'') with a symmetric tensor $\tilde{\mathbf{B}}_\beta$, the tensor $\mathbf{R}_\beta$ must satisfy the equation $\mathbf{B}_\beta-\mathbf{B}_\beta^{\mathrm{T}}=(\mathbf{T}_{\alpha\beta}-p_\alpha\mathbf{E})\,\mathbf{R}_\beta+\mathbf{R}_\beta(\mathbf{T}_{\alpha\beta}-p_\alpha\mathbf{E})$. In a system of coordinates formed by the eigenvectors of the tensor $\mathbf{T}_{\alpha\beta}$ we have $R_{\beta ij}=(B_{\beta ij}-B_{\beta ij})/(T_{\alpha\beta j}+T_{\alpha\beta j}-2p_\alpha)$. This expression must be definite for any deformation mode and, specifically, when $T_{\alpha\beta i}+T_{\alpha\beta j}=2p_\alpha$. We will determine the condition under which the general relaxation-type equation

$$\overset{0}{\mathbf{T}}+\varkappa\mathbf{T}+\mathbf{Y}=\varphi\mathbf{E}, \tag{20}$$

with the symmetric tensor Y, depending on the flow mode at a given point at instant of time t, becomes transformed into the system of Eqs. (17') and (17'') with a symmetric tensor B. Inserting $\mathbf{T}=\tilde{\mathbf{T}}-p\mathbf{E}$ into Eq. (20) yields

$$\overset{0}{\tilde{\mathbf{T}}}+\varkappa\tilde{\mathbf{T}}+\mathbf{Y}=0,\quad \frac{Dp}{Dt}+\varkappa p=-\varphi. \tag{21}$$

System (21) corresponds to the form (17'), (17'') if there is a tensor $\mathbf{B}$ which satisfies the equation $\mathbf{Y}=\mathbf{BT}+\mathbf{TB}$. In a system of coordinates formed by the eigenvectors of the tensor $\mathbf{T}$ we have $B_{ij}=Y_{ij}/(T_i+T_j)$. This relation must be definite for any deformation mode and, specifically, when $T_i+T_j=0$.

Therefore, not every integral equation (1) and relaxation equation (20) is reducible to the system (17'), (17'') with a symmetric tensor $\mathbf{B}_\beta$. Let us examine closer the meaning of symmetric tensors $\mathbf{B}_\beta$. From representation (11) and Eq. (14) we have

$$\frac{D\Omega_\beta}{Dt}=\boldsymbol{\omega}_\beta^{\mathrm{T}}(\mathbf{W}+\mathbf{B}_\beta)\,\boldsymbol{\omega}_\beta+\boldsymbol{\omega}_\beta^{\mathrm{T}}(-\mathbf{W}+\mathbf{B}_\beta)\,\boldsymbol{\omega}_\beta=2\boldsymbol{\omega}_\beta^{\mathrm{T}}\mathbf{B}_\beta\boldsymbol{\omega}_\beta. \tag{22}$$

It is evident here that $\Omega_\beta=\mathbf{E}$ and, consequently, no stresses appear when $\mathbf{B}_\beta=0$. When $\mathbf{B}_\beta=\mathbf{D}$, where $\mathbf{D}=\frac{1}{2}(\nabla\vec{v}+\nabla\vec{v}^{\mathrm{T}})$ is the strain rate tensor of the medium, then $\mathbf{A}_\beta=\nabla\vec{v}$ and Eq. (14) yields $w=\mathbf{F}t(\tau)$ as the relative strain gradient in the medium [1-3]. These properties as well as the transformation law for tensor

ω_β make it possible, upon a change of the reference system, to interpret tensor $\mathbf{B}_\beta$ as the strain rate tensor of a β-lattice. The quantities $\varkappa_\alpha$, φ_α, $\mathbf{B}_\beta$ in Eqs. (17') and (17'') depend on the flow characteristics at instant of time t, namely on tensors $\mathbf{D}$ and $\mathbf{T}_{\alpha\beta}$, internal parameters characterizing the medium, higher than first-order strain rate tensors (Rivlin–Ericksen, White–Metzner, Goddard [3, 10]), etc. Since Eqs. (17') and (17'') satisfy the principle of objectivity, these functions are isotropic [1, 2]. Using the formulas for representing isotropic scalar and vector functions of symmetric tensors [11, 12], one can write the expressions for $\varkappa_\alpha$, φ_α, $\mathbf{B}_\beta$ in a more explicit form. In the simplest case of these quantities depending only on the two tensors $\mathbf{D}$ and $\mathbf{T}_{\alpha\beta}$ [2, 3], for instance,

$$\mathbf{B}_\beta = m_1\mathbf{D} + m_2\mathbf{T}_{\alpha\beta} + m_3\mathbf{D}^2 + m_4\mathbf{T}^2_{\alpha\beta} + m_5(\mathbf{D}\mathbf{T}_{\alpha\beta} + \mathbf{T}_{\alpha\beta}\mathbf{D}) + m_6(\mathbf{D}^2\mathbf{T}_{\alpha\beta} + \mathbf{T}_{\alpha\beta}\mathbf{D}^2) + m_7(\mathbf{D}\mathbf{T}^2_{\alpha\beta} + \mathbf{T}^2_{\alpha\beta}\mathbf{D}) + m_8(\mathbf{D}^2\mathbf{T}^2_{\alpha\beta} + \mathbf{T}^2_{\alpha\beta}\mathbf{D}^2). \quad (23)$$

The isotropic term has been omitted in expression (23). Functions $\varkappa_\alpha$, φ_α, and also m_i (i = 1-8) depend on internal parameters and on nine invariants of tensors $\mathbf{D}$ and $\mathbf{T}_{\alpha\beta}$ (inasmuch as $\operatorname{tr}\mathbf{D} = 0$): $\operatorname{tr}\mathbf{D}^2$, $\operatorname{tr}\mathbf{D}^3$, $\operatorname{tr}\mathbf{T}_{\alpha\beta}$, $\operatorname{tr}\mathbf{T}^2_{\alpha\beta}$, $\operatorname{tr}\mathbf{T}^3_{\alpha\beta}$, $\operatorname{tr}(\mathbf{T}_{\alpha\beta}\mathbf{D})$, $\operatorname{tr}(\mathbf{T}_{\alpha\beta}\mathbf{D}^2)$, $\operatorname{tr}(\mathbf{D}\mathbf{T}^2_{\alpha\beta})$, $\operatorname{tr}(\mathbf{D}^2\mathbf{T}^2_{\alpha\beta})$. The isotropic tensor function $\mathbf{B}_\beta\mathbf{T}_{\alpha\beta} + \mathbf{T}_{\alpha\beta}\mathbf{B}_\beta$ in Eq. (17') can also be expressed in a form analogous to expression (23). Such a transformation is effected through sequential elimination of tensors $\mathbf{T}^3_{\alpha\beta}$, $\mathbf{T}_{\alpha\beta}\mathbf{D}\mathbf{T}_{\alpha\beta}$, $\mathbf{T}_{\alpha\beta}\mathbf{D}^2\mathbf{T}_{\alpha\beta}$ with the air of the Hamilton–Cayley theorem and the consequent identity for symmetric tensors

$$\mathbf{TAT} \equiv (\mathbf{TA} + \mathbf{AT})\operatorname{tr}\mathbf{T} - \mathbf{T}^2\mathbf{A} - \mathbf{A}\mathbf{T}^2 - \frac{1}{2}\mathbf{A}(\operatorname{tr}^2\mathbf{T} - \operatorname{tr}\mathbf{T}^2) + \mathbf{T}^2\operatorname{tr}\mathbf{A} +$$
$$+ \mathbf{T}(\operatorname{tr}\mathbf{TA} - \operatorname{tr}\mathbf{T}\operatorname{tr}\mathbf{A}) + \mathbf{E}\left[\operatorname{tr}\mathbf{T}^2\mathbf{A} - \operatorname{tr}\mathbf{T}\operatorname{tr}\mathbf{TA} + \frac{1}{2}(\operatorname{tr}^2\mathbf{T} - \operatorname{tr}\mathbf{T}^2)\operatorname{tr}\mathbf{A}\right].$$

For calculating the characteristics of a slow flow one can use differential rheological equations. For the generalized Maxwell fluid (17'), (17'') these equations are obtained by insertion of the strain tensor expansion

$$\Omega_\beta = \mathbf{E} + \sum_{n=1}^{\infty} \frac{\mathbf{V}_{\beta n}(\tau - t)^n}{n!}, \qquad \mathbf{V}_{\beta n} = \left.\frac{D^n\Omega_\beta}{D\tau^n}\right|_{\tau=t}$$

into integral Eqs. (18) and (19), respectively:

$$\mathbf{T} = \sum_{\alpha\beta}\sum_{n=1}^{\infty} q_{\alpha n}\mathbf{V}_{\beta n}, \quad q_{\alpha n} = \int_{-\infty}^{t} \varphi_\alpha \frac{(\tau - t)^n}{n!}\exp\left(-\int_\tau^t \varkappa_\alpha d\xi\right)d\tau,$$
$$\mathbf{T} = \sum_{\alpha\beta}\sum_{n=1}^{\infty} \tilde{q}_{\alpha, n-1}\mathbf{V}_{\beta n}, \quad \tilde{q}_{\alpha n} = -\int_{-\infty}^{t} \tilde{\varphi}_\alpha \frac{(\tau - t)^n}{n!}\exp\left(-\int_\tau^t \varkappa_\alpha d\xi\right)d\tau.$$

As $t \to -\infty$, we find that $\varphi_\alpha \to \varphi_{0,\alpha}$, $\tilde{\varphi}_\alpha \to \tilde{\varphi}_{0,\alpha}$, $\varkappa_\alpha \to \varkappa_{0,\alpha}$ with subscript 0 denoting the values of these quantities in the undeformed state and $q_{\alpha,n} \to (-1)^n \varphi_{0,\alpha}/\varkappa^{n+1}_{0,\alpha}$, $\tilde{q}_{\alpha n} \to (-1)^{n+1}\tilde{\varphi}_{0,\alpha}/\varkappa^{n+1}_{0,\alpha}$. When functions φ_α, $\tilde{\varphi}_\alpha$, $\varkappa_\alpha$ depend on the deformation mode, then the quantities $q_{\alpha n}$, $\tilde{q}_{\alpha,n}$ can be conveniently expressed in the form

$$q_{\alpha n} = \sum_{k=0}^{n} \frac{(-1)^k t^k \beta_{\alpha k}(t)}{k!(n-k)!}, \qquad \tilde{q}_{\alpha n} = -\sum_{k=0}^{n} \frac{(-1)^k t^k \tilde{\beta}_{\alpha k}(t)}{k!(n-k)!},$$

with the functions determined from the equations

$$\frac{D\beta_{\alpha k}}{Dt} + \varkappa_\alpha\beta_{\alpha k} = \varphi_\alpha t^k, \qquad \frac{D\tilde{\beta}_{\alpha k}}{Dt} + \varkappa_\alpha\tilde{\beta}_{\alpha k} = \tilde{\varphi}_\alpha t^k.$$

Using the multiplicativity relation (16), one can rewrite the expression for tensor $\mathbf{V}_{\beta n}$ in the form (t_0 is an arbitrary instant of time)

$$\mathbf{V}_{\beta n}(t) = \omega^{\mathrm{T}}_\beta(t, t_0)\frac{D^n\Omega_\beta(t_0, t)}{Dt^n}\omega_\beta(t, t_0).$$

Differentiating this expression with respect to t, we obtain the recurrence relation

$$\mathbf{V}_{\beta, n+1} = \frac{D\mathbf{V}_{\beta n}}{Dt} + \mathbf{A}^{\mathrm{T}}_\beta\mathbf{V}_{\beta n} + \mathbf{V}_{\beta n}\mathbf{A}_\beta, \quad \mathbf{V}_{\beta 0} \equiv \mathbf{E},$$

analogous to the recurrence relations for Rivlin–Ericksen and White–Metzner tensors [3]. Specifically we have $\mathbf{V}_{\beta 1} = 2\mathbf{B}_\beta$. When the tensor $\mathbf{B}_\beta$ depends on the stress tensor, then the derivative $D\mathbf{V}_{\beta n}/Dt$ must be calculated by elimination of the derivatives of the stress tensor with the aid of Eq. (17').

The preceding mathematical analysis thus reveals that most rheological equations for flowing polymer media can be derived from the equations of a Maxwell fluid, namely by introduction of the dependence of quantities $\varkappa_\alpha$, φ_α on the flow characteristics and by use of more intricate "effective" strain rate tensors $\mathbf{B}_\beta$ in

Eqs. (17') and (17''). Into account is also taken here that the tensors $\mathbf{B}_\beta$ can depend on the flow characteristics.

LITERATURE CITED

1. C. Truesdell and W. Noll, "Nonlinear field theories of mechanics," in: Handbook of Physics, Vol. III/3, Springer-Verlag, Berlin (1965).
2. C. Truesdell (ed.), Continuum Mechanics, Vol. 2, Rational Mechanics of Materials, Gordon and Breach (1965).
3. G. Astarita and G. Marucci, Basic Hydromechanics of Newtonian Fluid [Russian translation], Mir, Moscow (1978).
4. A. S. Lodge, Body Tensor Fields in Continuum Mechanics, Academic Press, New York (1974).
5. R. R. Bird, R. C. Armstrong, and O. Hassager, Dynamics of Polymer Fluids, Vol. 1, Wiley, New York (1977),
6. G. V. Vinogradov and A. Ya. Malkin, Rheology of Polymers [in Russian], Khimiya, Moscow (1977).
7. Chaig Dei Khan, Rheology of Polymer Reprocessing [in Russian], Khimiya, Moscow (1979).
8. A. S. Lodge, "Constitutive equations from molecular network theories for polymer solutions," Rheol. Acta, 7, No. 4, 379-392 (1968).
9. F. R. Gantmakher, Theory of Matrices, Chelsea Publ.
10. T. D. Goddard, "Polymeric fluid mechanics," Adv. Appl. Mech., 19, 143-219 (1979).
11. C. Truesdell and R. A. Toupin, "Classical field theories," Handbook of Physics, Vol. III/1, Springer-Verlag, Berlin (1960), pp. 226-793.
12. L. I. Sedov, Introduction to Mechanics of Continuous Media [in Russian], Fizmatgiz, Moscow (1962).

INTERRELATIONSHIP OF RHEOLOGICAL AND BIOLOGICAL CHARACTERISTICS IN COMPLEX SYSTEMS

É. G. Tutova, É. V. Ivashkevich, and I. V. Zhavnerko

UDC 532.535

The interrelationship of rheological properties and quantitative indices of fluids of biological origin is analyzed. Possible variants of the use of the viscosity for estimating the state of the system are presented.

In studying labile systems, whose properties depend greatly on the parameters of the external medium (t, P, φ) or the state of the system itself (t, W), it is necessary to choose a characteristic physical indicator, which reflects to a certain extent the state of the substance, as well as the kinetics and dynamics of its variation. Typical representatives of such materials are heterogeneous systems of biological origin – microbe biomasses. It is well known that the presently existing methods of microbiological analysis are imperfect and are distinguished by their long duration, measured in days, and high degree of error. The effect of the error can be eliminated by multiple repetition of the experiment and statistical analysis of the results obtained, as is customary in studying probabilistic processes. However, in this case, the duration of the analysis increases even more, which can be eliminated only by developing and applying new improved methods, based on the interrelationship of physical and biological properties of the system.

It is well known that microbiological materials of different nature are characterized by a wide range of rheological properties from Newtonian to plastic [1], which can serve as qualitative and quantitative indices of heat and mass transfer in bioengineering and biotechnology processes. Thus, the viscosity of the starting feed media exceeds by not more than a factor of 1.5-2 the viscosity of water, while during the growth of life of microorganisms, this quantity increases by one to two orders of magnitude. The increase in the viscosity of the

A. V. Lykov Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk.
Translated from Inzherno-Fizicheskii Zhurnal, Vol. 42, No. 4, pp. 677-681, April, 1982. Original article submitted February 5, 1981.